Sets, Relations and Functions

Sets in Maths

A set is a collection of well-defined objects. The objects of a set are taken as distinct only on the ground of simplicity.

A set of sets is frequently called a family or collection of sets. For example, suppose we have a family of sets consisting of A_1, A_2, A_3,….. up to A_n, that is the family {A_1, A_2, A_3,….., A_n } and can be denoted as

S = {A_i | i belongs to N and $1 \leq i \leq n$}.

Notation: A set is denoted by a capital letter and represented by listing all its elements between curly brackets, such as { }.

Types of Sets

Singleton set

A set contains only one element. For example, A = {3} and B = {pencil}. Here, A and B are containing only one element, so both are singleton sets.

Empty Set/Null Set

An empty set is a set with no element. It is denoted by A = { } or A = ϕ.

Proper set

If A and B are two sets, then A is a proper subset of B if $A \subseteq B$, but $A \neq B$.

For example, if B = {2, 3, 5}, then A = {2, 5} is a proper subset of B.

Power Set

The collection of all subsets of a set is the power set of that set. If A is the set, then P(A) is denoted as its power set.

The number of elements contained by any power set can be calculated by n[P(A)] = $2^{n,}$ where n is the number of elements in set A.

For example, if A = {1, 2}, then P(A) = {∅, {1}, {2}, {1, 2}}

Number of elements in P(A) = 2^2 = 4

Finite Set

A set contains a finite number of elements. For example, A = { 2, 4, 6, 8, 10} and B = { a, v, t}. There are 5 objects in set A and 3 elements contained in set B.

Infinite set

If the number of elements in a set is infinite, the set is called an infinite set. For example, N = set of whole numbers = { 0, 1, 2, 3, 4, 5, ……}.

Universal Set

Any set, which is a superset of all the sets under consideration, and usually it is denoted as S or U.

For example, Let P = {3, 4, 7} and Q = {1, 2, 3}, then we take S = {1, 2, 3, 4, 7} as a universe set.

Equal Sets

Two sets, P and Q, are equal if both are a subset of each other.

Mathematically, if P ⊆ Q and Q ⊆ P, then P = Q.

For example, P = {3, 6, 8} and Q = {6, 3, 8}

Here, P and Q have exactly the same elements. Satisfy the condition P ⊆ Q and Q ⊆ P.

Thus, P = Q.

Operations on Sets

In sets theory, there are basically three operations applicable to two sets, and they are as follows:

- Union of two sets
- Intersection of two sets

- Difference of two sets

Relations in Maths

A relation is helpful in finding the relationship between **input** and **output** of a function.

A relation R, from a non-empty set P to another non-empty set Q, is a subset of P X Q.

For example, let P = {a, b, c} and Q = {3, 4} and

Let R = {(a, 3), (a, 4), (b, 3), (b, 4), (c, 3), (c, 4)}

Here, R is a subset of A x B. Therefore, R is a relation from P to Q.

Types of Relations

The relationship between the elements of different pairs of sets is termed as relations between the sets.

The set consisting of all the x-values is called the domain while the set consisting of all the y-values is called the range.
For example- The following table represents the x-value and y-value in separate columns.

X	Y
1	6
-3	2
5	0
-1	-5
4	2

After pairing each pair in the same order as they are listed.

Ordered Pair = {(1 , 6) , (-3 , 2) , (5 , 0) , (-1 , -5) , (4 , 2)}

Domain= {1 , -3 , 5 , -1 , 4}

Range= {6 , 2 , 0 , -5 , 2}

Empty Relation

An empty relation (or void relation) is one in which there is no relation between any elements of a set. For example, if set A = {1, 2, 3}, then one of the void relations can be R = {x, y} where, |x – y| = 8. For an empty relation,

$R = \varphi \subset A \times A$

Universal Relation

A universal (or full relation) is a type of relation in which every element of a set is related to each other. Consider set A = {a, b, c}. Now, one of the universal relations will be R = {x, y}, where |x – y| ≥ 0. For a universal relation,

$R = A \times A$

Identity Relation

In an identity relation, every element of a set is related to itself only. For example, in a set A = {a, b, c}, the identity relation will be I = {a, a}, {b, b}, {c, c}. For an identity relation,

$I = \{(a, a), a \in A\}$

Inverse Relation

Inverse relation is seen when a set has elements which are inverse pairs of another set. For example, if set A = {(a, b), (c, d)}, then inverse relation will be R^{-1} = {(b, a), (d, c)}. So, for an inverse relation,

$R^{-1} = \{(b, a): (a, b) \in R\}$

Reflexive Relation

In a reflexive relation, every element maps to itself. For example, consider a set A = {1, 2,}. Now, an example of reflexive relation will be R = {(1, 1), (2, 2), (1, 2), (2, 1)}. The reflexive relation is given by, (a, a) ∈ R

Symmetric Relation

In a symmetric relation, if a = b is true, then b = a is also true. In other words, a relation R is symmetric only if (b, a) ∈ R is true when (a,b) ∈ R. An example of symmetric relation will be R = {(1, 2), (2, 1)} for a set A = {1, 2}. So, for a symmetric relation,

aRb ⇒ bRa, ∀ a, b ∈ A

Transitive Relation

For transitive relation, if (x, y) ∈ R, (y, z) ∈ R, then (x, z) ∈ R. For a transitive relation,

aRb and bRc ⇒ aRc ∀ a, b, c ∈ A

Equivalence Relation

If a relation is reflexive, symmetric and transitive at the same time, it is known as an equivalence relation.

Equivalence Relation Definition

A relations in maths for real numbers R defined on a set A is said to be an equivalence relation if and only if it is reflexive, symmetric and transitive. They are often used to group together objects that are similar, or equivalent. It satisfies the following conditions for all elements a, b, c ∈ A:

- **Reflexive -** R is reflexive if (a, a) ∈ R for all a ∈ A
- **Symmetric -** R is symmetric if and only if (a, b) ∈ R ⇒ (b, a) ∈ R for all a, b ∈ A
- **Transitive -** R is transitive if and only if (a, b) ∈ R and (b, c) ∈ R ⇒ (a, c) ∈ R for all a, b, c ∈ A

Partial Order Relations

A relation R on a set A is called a partial order relation if it satisfies the following three properties:

1. Relation R is Reflexive, i.e. aRa ∀ a∈A.
2. Relation R is Antisymmetric, i.e., aRb and bRa ⟹ a = b.
3. Relation R is transitive, i.e., aRb and bRc ⟹ aRc.

Example1: Show whether the relation (x, y) ∈ R, if, x ≥ y defined on the set of +ve integers is a partial order relation.

Solution: Consider the set A = {1, 2, 3, 4} containing four +ve integers. Find the relation for this set such as R = {(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3), (1, 1), (2, 2), (3, 3), (4, 4)}.

Reflexive: The relation is reflexive as for every a ∈ A. (a, a) ∈ R, i.e. (1, 1), (2, 2), (3, 3), (4, 4) ∈ R.

Antisymmetric: The relation is antisymmetric as whenever (a, b) and (b, a) ∈ R, we have a = b.

Transitive: The relation is transitive as whenever (a, b) and (b, c) ∈ R, we have (a, c) ∈ R.

Example: (4, 2) ∈ R and (2, 1) ∈ R, implies (4, 1) ∈ R.

As the relation is reflexive, antisymmetric and transitive. Hence, it is a partial order relation.

Empty Sets:

Empty Sets are sets with no items or elements in them and is also called null set. The empty set is represented by the symbol ∅ = { }. It is pronounced 'phi'. Set X = {} as an example. It is also known as a void set or a null set.

A set which does not contain any element is called the **empty set or the null set** or the void set. For example, the set of the number of outcomes for getting a number greater than 6 when rolling a die. As we know, the outcomes of rolling a die are 1, 2, 3, 4, 5, and 6. Thus, the set with numbers greater than 6 here will be { }. That means there will be no elements and is called the empty set.

Empty Set Symbol

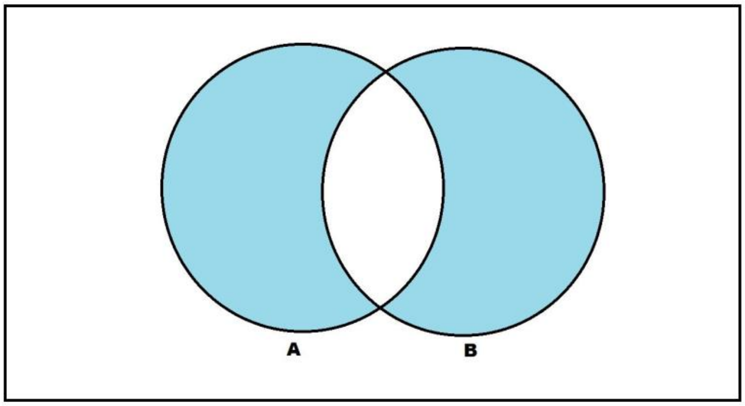
The **empty set is denoted by** the symbol "Φ" and "φ" or { }.

Cardinality

Cardinality refers to the number that is obtained after counting something. Thus, the cardinality of a set is the number of elements in it. For example, the set {1, 2, 3, 4, 5} has cardinality five which is more than the cardinality of {1, 2, 3} which is three.

Symmetric difference

In mathematics, the symmetric difference of two sets, also known as the disjunctive union and set sum, is the set of elements which are in either of the sets, but not in their intersection.



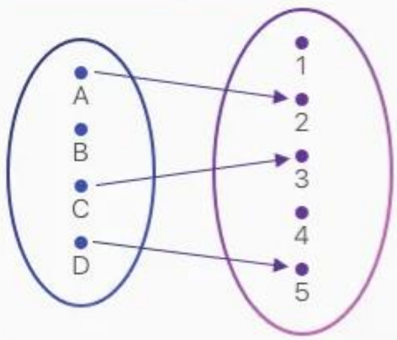
For an example of the symmetric difference, we will consider the sets A = {1,2,3,4,5} and B = {2,4,6}. The symmetric difference between these sets is {1,3,5,6}.

The symmetric difference between two sets can also be expressed as the union of two sets minus the intersection between them -
A Δ B = (A ∪ B) - (A ∩ B)

FUNCTION

A *function* in set theory world is simply a mapping of some (or all) elements from Set A to some (or all) elements in Set B. In the example above, the collection of all the possible elements in A is known as the ***domain****;* while the elements in A that act as inputs are specially named ***arguments***. On the right, the collection of all possible outputs (also known as “range” in other branches), is referred to as the ***codomain***; while the collection of actual output elements in B mapped from A is known as the ***image.***

Set Function Definitions I



Domain	{ A,B,C,D }
Arguments	{ A,C,D }
Codomain	{ 1,2,3,4,5 }
Image	{ 2,3,5 }

A function or mapping (Defined as f: X → Y) is a relationship from elements of one set X to elements of another set Y (X and Y are non-empty sets). X is called Domain and Y is called Codomain of function ‘f’.

Function ‘f’ is a relation on X and Y such that for each x ∊ X, there exists a unique y ∊ Y such that (x,y) ∊ R. ‘x’ is called pre-image and ‘y’ is called image of function f.

A function can be one to one or many to one but not one to many.

Injective / One-to-one function

A function f: A → B is injective or one-to-one function if for every b ∊ B, there exists at most one a ∊ A such that f(s) = t.

This means a function **f** is injective if $a_1 \neq a_2$ implies f(a1) ≠ f(a2).

Example

- f: N → N, f(x) = 5x is injective.
- f: N → N, f(x) = x^2 is injective.
- f: R → R, f(x) = x^2 is not injective as $(-x)^2 = x^2$

Surjective / Onto function

A function f: A → B is surjective (onto) if the image of f equals its range. Equivalently, for every b ∊ B, there exists some a ∊ A such that f(a) = b. This means that for any y in B, there exists some x in A such that y = f(x).

Example

- f : N → N, f(x) = x + 2 is surjective.
- f : R → R, f(x) = x^2 is not surjective since we cannot find a real number whose square is negative.

Bijective / One-to-one Correspondent

A function f: A → B is bijective or one-to-one correspondent if and only if **f** is both injective and surjective.

Problem

Prove that a function f: R → R defined by f(x) = 2x – 3 is a bijective function.

Explanation − We have to prove this function is both injective and surjective.

If $f(x_1) = f(x_2)$, then $2x_1 – 3 = 2x_2 – 3$ and it implies that $x_1 = x_2$.

Hence, f is **injective**.

Here, 2x – 3 = y

So, x = (y+5)/3 which belongs to R and f(x) = y.

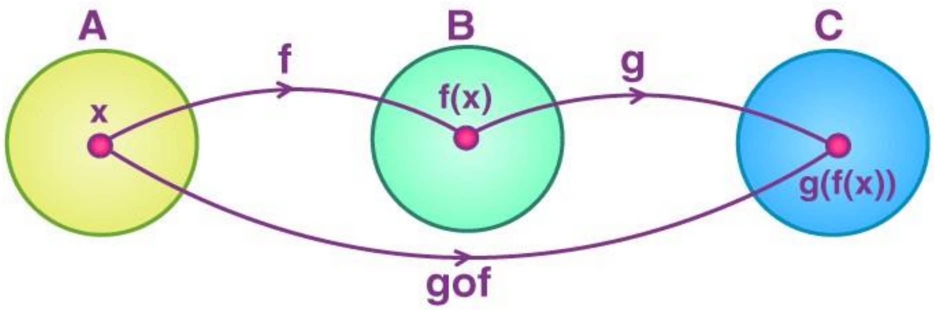
Hence, f is **surjective**.

Since **f** is both **surjective** and **injective**, we can say **f** is **bijective**.

Composite Functions Definition

Let f : A → B and g : B → C be two functions. Then the composition of f and g, denoted by g ∘ f, is defined as the function g ∘ f : A → C given by g ∘ f (x) = g(f (x)), ∀ x ∈ A.

The below figure shows the representation of **composite functions**.



Graph of Signum Function

Signum function simply yields the sign for the assigned values of x. For x value higher than zero, the value assigned to the output is +1, for x value lesser than zero, the value assigned to the output is -1, and for x value equal to zero, the output is equivalent to zero.

Signum Function

The signum function f: R→R represented by:

$$f(x) = x \quad x > 0$$

$$= 0 \quad x = 0$$

$$= -x \quad x < 0$$

OR

$$f(x) = \frac{|x|}{x}, if\ x \neq 0$$
$$= 0,\ if\ x = 0$$

Domain and Range of Signum Function

The domain of the signum function covers all the real numbers and is represented along the x-axis, and the range of the signum function has simply two values, +1, -1, drawn on the y-axis.

Domain = R

Range = {-1, 0, 1}

Properties of Signum Function

Let us consider x. The function $sgn(x)$ yielding a real number, is defined by:

$$sgn(x) = \begin{cases} 1 & \text{if } 0 < x \\ -1 & \text{if } x < 0 \\ 0 & \text{otherwise} \end{cases}$$